ПРОБЛЕМЫ МЕХАНИКИ И МАШИНОСТРОЕНИЯ

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On the contact pressure distribution during the sphere indentation into an elastic-plastic half-space

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The paper provides a justification for the approach to determining the pressure distribution function on the contact surface when a spherical indenter is introduced into an elastic-plastic half-space. It is shown that the use of the Hertz theory to describe the process of unloading a spherical indenter is an approximate approach, since the contact of two bodies initially in contact at a point is reduced to the contact of two second-order paraboloids. This leads to the implicit assumption that a "Hertzian" pressure distribution takes place before unloading. A conceptual model of the process of kinetic indentation of a sphere is proposed, according to which, in the process of loading a sphere, as a result of elastic and plastic deformation processes, a pressure distribution is formed in the contact between the sphere and the material, which is determined by the equation of the loading curve. When the sphere is unloaded, the magnitude of the displacements of the restored surface is equal to the displacements of the restored elastic half-space after the action of the desired pressure distribution on it. Under repeated loading of the sphere, the displacements of the restored surface will be equivalent to the displacements of the elastic half-space under the action of the desired pressure distribution on it. Equations are obtained for the distribution of contact pressure when the sphere is introduced into an elastic-plastic half-space.

Keywords: spherical indenter, elastic-plastic half-space, contact pressure distribution, loading curve, unloading curve.

1. Introduction

Historically, sphere indentation testing of materials at the beginning of the last century was used to determine their hardness [1, 2]. Obviously, the appearance of these works was influenced by the concept of absolute hardness of G. Hertz, who in 1881 solved the problem of elastic contact between a ball and a plane. Since the middle of the last century, studies began to appear on the determination of the elastic modulus and other mechanical properties [3–9], which led to the emergence of the kinetic indentation method. The method for determining the modulus of elasticity is based on the Bulychev – Alekhine – Shorshorov (BASh) equation for the stiffness of the initial part of the unloading curve, which has the form

$$S = \left.\frac{dP_e}{dh_e}\right|_{h_e = h_m} = \frac{2\sqrt{A}}{\sqrt{\pi}} \cdot E^*, \tag{1}$$

where A is the projection of the contact area of the indenter with the material, E^* is the reduced modulus of elasticity;

$$E^* = \left(\frac{1-\nu^2}{E} + \frac{1-\nu_i^2}{E_i}\right)^{-1}, \tag{2}$$

ν, ν_i, and E, E_i, are Poisson's ratios and elastic moduli of the material and indenter.

A detailed analysis of the elastic indentation of axisymmetric indenters is presented by the authors of [10]. For practical purposes, the stiffness of the initial part of the unloading curve is represented by the expression

$$S = 2E^* a, \tag{3}$$

where a is the radius of the contact area.

However, as noted in our recent work [11], due to a certain dissatisfaction in determining the modulus of elasticity and the radius of the contact area, various corrective and correction factors and parameters are introduced into Eq. (3) [12-18], for example, in [12]

$$S = \beta_c 2E^* a, \tag{4}$$

where β_c is the correction factor.

As a result of the research, for the rigidity of the initial part of the unloading curve, the authors of [11] proposed the following expression

$$S = \frac{\pi\gamma}{K_\beta} E^* a, \tag{5}$$

where γ is the exponent of the unloading curve

$$\gamma = \gamma(\bar{a}) = \frac{1 + 2\sqrt{1-\bar{a}^2}}{1 + \sqrt{1-\bar{a}^2}}, \tag{6}$$

$\bar{a} = a/R$, R is the radius of the spherical indenter, for small values the value $\gamma \approx 1.5$;

$$K_\beta = K_\beta(\beta) = 2^{2\beta+1}(1+\beta)\mathrm{B}(\beta+1,\beta+1), \qquad (7)$$

$\mathrm{B}(x_1, x_2)$ is the beta function: $\beta+1=\alpha$, α is the exponent in the load curve equation

$$P(h) = C_l h^\alpha. \qquad (8)$$

In its physical essence, the parameter β is the exponential in the equation of pressure distribution $p(r)$ on the contact surface in the loaded state of the indenter during its elastoplastic penetration

$$p(r) = p_0(1 - r^2/a^2)^\beta, \qquad (9)$$

where $\beta=\alpha-1$, $p_0 = p_m(\beta+1)$, $p_m = P/(\pi a^2)$ is the average pressure in the contact.

For the Hertzian contact pressure distribution: $\beta = 0.5$, $K_\beta = 3\pi/4$, $S = 2E^* a$; for the uniform pressure: $\beta = 0$, $K_\beta = 2$, $S = (3\pi/4)E^* a = 2.3562 \cdot E^* a$.

It should be noted that Eq. (5) for the stiffness of the initial part of the unloading curve was previously obtained in [19] as a result of determining the corrective factor for Eq. (4).

The purpose of this work is a more detailed substantiation of the approach to determining the pressure distribution function on the contact surface during the introduction of a spherical indenter, since in [11] to explain this issue, the authors limited themselves to referring to earlier studies of the team.

Thus, the present work is an addition to the research [11].

2. Methodological aspects

The Hertz elastic contact theory is widely used in the technical literature to describe the process of indenter unloading and reloading. As an example, we can cite well-known monographs on the mechanics of contact interaction [20-22], monographs on the diagnosis of mechanical properties of materials [23, 24], the work of researchers at the National Research University "MEI" [25, 26], Chinese researchers [27, 28]. For all the above publications, the use of the "Hertzian" contact pressure distribution (Eq. (9) at $\beta = 0.5$) is characteristic. This result can be explained as follows. The Hertz 's theory assumes the following assumptions: 1) the surfaces of the bodies are smooth and inconsistent (by definition [20]; 2) the characteristic size of the contact area *a* << *R*, where *R* is the relative radius of curvature; 3) deformations are small; 4) each of the contacting bodies is considered as an elastic half-space; 5) there is no friction.

The contact of two elastic bodies initially touching at a point is considered in detail in [29, p. 74]. First, a system of Cartesian coordinates is introduced, associated with a common tangent plane to the surfaces of elastic bodies at the point of their contact. After the expansion of the functions describing the contacting surfaces, according to the Maclaurin formula and neglecting the terms of a higher order of smallness, we obtain the contact of their second-order paraboloids, since the expansion begins with quadratic terms.

Thus, the application of the Hertz theory to the elastic contact during repeated loading of the indenter is reduced to the contact of two second-order paraboloids with relative curvature

$$\frac{1}{R^*} = \frac{1}{R} - \frac{1}{R_0}, \qquad (10)$$

where R is the radius of the spherical indenter, R_0^{-1} is the curvature of the reconstructed hole, usually determined from the radius and depth of the residual hole.

For such a contact, the pressure distribution is described by Eq. (9) at $\beta = 0.5$, from which the normal displacements are found [20], and then the radius of the contact area

$$a = \frac{\pi p_0 R}{2E^*}, \qquad (11)$$

and the mutual rapprochement of two bodies

$$\delta = \frac{\pi a p_0}{2E^*}. \qquad (12)$$

Full compressive load

$$P = \frac{2}{3} p_0 a^2. \qquad (13)$$

To solve the practical problems, it is convenient to set a full load and determine the parameters *a*, δ and p_0 by the Hertz equations for second-order paraboloids obtained from (11) - (13).

Regarding the use of the Hertz theory to describe the unloading process in [20, p. 210], it is indicated that this approach is approximate, since it was implicitly assumed that a "Hertz" pressure distribution takes place before unloading, and the reconstructed profile is therefore a circular arc. In fact, the pressure distribution is more uniform than the "Hertz" distribution.

A similar result was obtained by the authors [30, p. 69], who point to their verification of the assumption in [3] that the nature of pressure distribution at the print site does not depend on the properties of the material and the degree of deformation in the print. The results of the experiment showed that this assumption in determining the modulus of elasticity can change its true value by 10 ... 20 %. The real distribution of pressure is between the uniform distribution and the distribution over the hemisphere, realized with the elastic indentation of the ball.

It is also shown in [30, p. 15] that the radius of curvature on the surface of the reconstructed well c can differ almost twice, and in the center and on the periphery of the well the radius of curvature is greater than between them. This refutes the hypothesis that the reconstructed profile is an arc of a circle and reduces the reliability of the results when using the Hertz theory to describe the unloading process, which is typical for the above works [25, 26]. Moreover, when determining the modulus of elasticity in these works, a single indenter insertion with fixation of the applied force, the maximum depth of insertion and the depth of the restored well is sufficient. In [25], the data of the loading curve equation are given, but they are not used in determining the modulus of elasticity.

The Sneddon analysis was used in studies [27, 28] [5]. The introduction of a spherical indenter into a spherical well is described by the equation

$$h = \frac{a}{2}\left(\log\frac{R+a}{R-a} - \log\frac{R_0+a}{R_0-a}\right). \quad (14)$$

Decomposing expression (14) into a Taylor series and discarding the terms of the higher order of smallness, as in [29], we obtain the expression for the contact of two second-order paraboloids with an equivalent radius (10) and a reduced modulus of elasticity

$$E^* = \frac{S}{2\sqrt{hR^*}}. \quad (15)$$

Here, the depth of indenter insertion and the stiffness of the initial part of the unloading curve are determined from the kinetic indentation diagram. The modulus of elasticity of the material is found from Eq. (2).

The equalization of the pressure distribution on the contact surface during the elastic-plastic insertion of indenters is indicated in [12] when presenting the concept of the "effective form of the indenter" for conical and pyramidal indentors, although the authors use the Hertz theory to determine the rigidity of the unloading curve. The authors argue that from a fundamental point of view, the shape of an effective indenter can be estimated by making simple assumptions about the distribution of pressure under the indenter. The pressure arising during loading is determined by complex processes of elastic and plastic deformation, which, as a rule, cannot be analyzed in a closed form. However, during unloading, the pressure decreases only due to elastic processes, and the shape of the print changes, forming a curved surface. Under repeated loading, this pressure distribution should be restored only by elastic processes. Thus, the pressure distribution at maximum load serves to link elastic-plastic processes during initial loading with elastic processes during unloading and overloading. In this context, the shape of an effective indenter should be such that it provides the same pressure distribution due to elastic deformation of a flat elastic half-space. To implement these ideas, it is necessary to know the actual distribution of pressure. Finite element modeling of the indentation of a conical indenter into elastic-plastic materials has shown that, in the first approximation, the pressure is evenly distributed. This is because plasticity tends to reduce the influence of the elastic feature at the tip of the cone and distribute the pressure more evenly.

Obviously, the above statement is also suitable for the introduction of spherical indenters. Taking into account the assumptions of the authors [20, 30] about the real pressure distribution between the uniform and the “Hertz", we will look for the real pressure distribution in the form (9), where $\beta = 0\ldots0.5$.

In classical monographs on the contact mechanics and the theory of elasticity [20, 31, etc.], only special cases of the stress-strain state for a uniform pressure distribution (β=0) and for the “Hertz” pressure (β=0.5.) are considered. For analytical expressions for engineering calculations of the stress-strain state under the action of a load of the form (9) on the half-space, the authors [32]. The reliability of the general solutions given in [32] for describing the stress-strain state is evidenced by their coincidence with the particular solutions obtained in [20, 31], as well as their use in describing the contact geometry in [33] with subsequent comparison with the results of finite element modeling.

In connection with the above, the following conceptual model of the process of instrumental indentation of a rigid sphere is proposed. During loading of the indenter, as a result of complex processes of elastic and plastic deformations in contact of the indenter with the material, a pressure distribution is formed, which is determined by Eq. (8) of the loading curve. When unloading the indenter, the magnitude of the vertical and horizontal displacements of the surface being restored is equal to the vertical and horizontal displacements of the elastic half-space being restored after the desired pressure distribution acts on it [34]. When the indenter is reloaded, the displacements of the restored surface will be equivalent to the corresponding displacements of the elastic half-space under the action of the desired pressure distribution on it.

3. Determination of the pressure distribution on the contact surface

The problem of determining the contact pressure distribution function was first posed in [35], where the indentation of a rigid ball into an elastoplastic half-space is considered. In this case, the method of experimental theoretical equilibrium developed by the author is used. One of the shortcomings of the work is that it did not take into account the “pile-up/sink-in” effects (Fig. 1) associated with the plastic extrusion of the material of the half-space around the sphere (heap formation) and the elastic punching of the half-space.

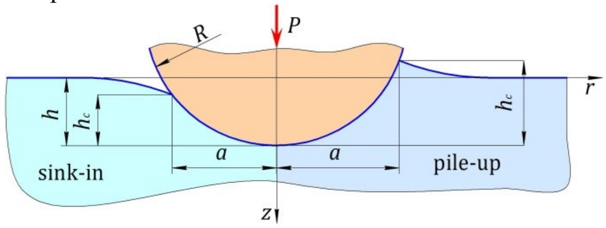

Fig.1. The effects of "pile-up/sink-in" when indentation the sphere

Let's consider this question in more detail. Using the technique [35], we consider that the indentation diagram is determined experimentally and is given by the power function (8).

The equation of forces balance

$$P = 2\pi\int_0^a p(r) r dr, \quad (16)$$

where $p(r)$ is the contact pressure distribution function.

The radius of the contact area for a spherical indenter is determined by the equation

$$a = \sqrt{2Rh_c - h_c^2}, \quad (17)$$

where h_c is the depth at which the sphere contacts the material of the half-space. Consider first the case when $R >> h_c$. As a result, we obtain the contact radius for the second-order paraboloid

$$a = \sqrt{2Rh_c}\,. \tag{18}$$

In [35] $a = \sqrt{2Rh}$.

For the function of a contact displacement

$$u(r) = \frac{a^2 - r^2}{2R} = h_c - \frac{r^2}{2R} = c^p h - \frac{r^2}{2R}\,. \tag{19}$$

where $c^p = h_c/h_0$. When $c^2 < 1$ there is a "sink-in" effect, when $c^2 > 1$ there is a "pile-up" effect.

Differentiating Eq. (19) with respect to r, we have

$$\frac{du}{dr} = -\frac{r}{R}, \qquad rdr = -Rdu\,. \tag{20}$$

The equation of forces balance, expressed through the function of contact displacements, has the form:

$$P = 2\pi R \int_0^{u_0} p(u)du\,. \tag{21}$$

where $u_0 = u(0) = h_c = c^p h$.

We will seek the pressure function in the form of a power function

$$p(u) = \xi u^\beta\,, \tag{22}$$

where ξ, β are two parameters of the desired pressure function.

Substituting (22) into (21), taking into account (8), we have

$$C_l h^\alpha = 2\pi R \xi \frac{(c^p h)^{\beta+1}}{\beta+1}\,. \tag{23}$$

The resulting equation can be satisfied under the condition

$$\alpha = \beta + 1\,, \qquad \beta = \alpha - 1; \tag{24}$$

$$\xi = \frac{C_l(\beta+1)}{2\pi R (c^p)^{\beta+1}} = \frac{C_l \alpha}{2\pi R (c^p)^\alpha}\,. \tag{25}$$

Substituting (24) and (25) into (22) we get

$$p(u) = \frac{C_l \alpha}{2\pi R (c^p)^\alpha} u^{\alpha-1}\,, \tag{26}$$

and taking into account (19)

$$p(r) = p_0(1 - r^2/a^2)^\beta\,, \tag{27}$$

where

$$p_0 = \frac{C_l \alpha h^\beta}{2\pi R c^p} = \frac{C_l h^\alpha (1+\beta)}{\pi 2 R c^p h} = \frac{P(1+\beta)}{\pi a^2} = p_m(1+\beta)\,. \tag{28}$$

p_m is the mean pressure at the contact area.

Since it is in the range from 1 to 1.5, then $\beta = 0\ldots0{,}5$.

Eqs. (27) and (28) were obtained for a parabolic indenter, for which the radius of the contact area is determined by the equation (18), or in the dimensionless form

$$\bar{a}_p = a_p/R = \sqrt{2\bar{h}_c}\,, \tag{29}$$

where $\bar{h}_c = h_c/R$

Similarly, from Eq. (17) for a spherical indenter, we have

$$\bar{a}_s = \sqrt{2\bar{h}_c - \bar{h}_c^2} = \sqrt{2\bar{h}_c}\left(\sqrt{1 - \bar{h}_c/2}\right) = \bar{a}_p\sqrt{1 - \bar{a}_p^2/4}\,. \tag{30}$$

If we use the solution (28) for a spherical indenter, then it should be taken into account that in order to ensure the forces balance, the mean pressure at the contact area must be equal to

$$p_m^s = p_m^p / \left(1 - \bar{a}_p^2/4\right). \tag{31}$$

Further, two extreme variants of contact pressure distribution models are possible:

a) the maximum pressure in the contact varies similarly to Eq. (31)

$$p_0^s = p_0^p / \left(1 - \bar{a}_p^2/4\right), \tag{32}$$

then

$$\beta_s = \beta_p = \alpha - 1\,; \tag{33}$$

b) maximum contact pressure $p_0^s = p_0^p$, then

$$1 + \beta_s = (1 + \beta_p)\left(1 - \bar{a}_p^2/4\right) = \alpha \cdot \left(1 - \bar{a}_p^2/4\right). \tag{34}$$

The first model of contact pressure distribution is simpler and more convenient to use, the second model equalizes the pressure even more, and as the value increases, a negative value is possible. Additional studies are needed to refine the model.

4. Conclusion

1. It is shown that the use of the Hertz theory to describe the process of unloading a spherical indenter is an approximate approach, since according to the Hertz theory the contact of two bodies initially in contact at a point is reduced to the contact of two paraboloids of the second order. Hence the implicit assumption that a "Hertzian" distribution of pressures takes place before unloading and that the reconstructed profile is an arc of a circle.

2. The Sneddon's analysis [5] for the contact of two spherical bodies leads to the same result (contact of two paraboloids of the second order).

3. A conceptual model of the process of instrumental indentation of a rigid sphere is proposed, according to which, during the loading of the sphere, as a result of complex processes of elastic and plastic deformations in contact of the sphere with the material, a pressure distribution is formed, which is determined by the equation of the loading curve. When unloading the sphere, the magnitude of the displacements of the surface being restored is equal to the displacements of the elastic half-space being restored after the desired pressure distribution acts on it. Under repeated loading of the sphere, the displacements of the restored surface will be equivalent to the corresponding displacements of the elastic half-space under the action of a desired pressure distribution on it.

4. Using the method of experimental theoretical equilibrium [35], Eqs. (27) and (28) for the distribution of contact pressure when the sphere is embedded in an elastic-plastic half-space are obtained.

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